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Regularized Lagrangian duality for linearly constrained quadratic optimization and trust-region problems

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Abstract In this paper we first establish a Lagrange multiplier condition characterizing a regularized Lagrangian duality for quadratic minimization problems with finitely many linear equality and quadratic inequality constraints, where the linear constraints are not relaxed in the regularized Lagrangian dual. In particular, in the case of a quadratic optimization problem with a single quadratic inequality constraint such as the linearly constrained trust-region problems, we show that the Slater constraint qualification (SCQ) is necessary and sufficient for the regularized Lagrangian duality in the sense that the regularized duality holds for each quadratic objective function over the constraints if and only if (SCQ) holds. A new theorem of the alternative for systems involving both equality constraints and two quadratic inequality constraints plays a key role. We also provide classes of quadratic programs, including a class of CDT-subproblems with linear equality constraints, where (SCQ) ensures regularized Lagrangian duality.

Keywords Quadratic nonconvex optimization · Regularized Lagrangian · Strong duality · Quadratic constraints · Linear equality constraints · Trust-region problems · Alternative theorems

Mathematics Subject Classification (2000) 90C26 · 90C46 · 90C20 · 90C30

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1 Introduction

Consider the quadratic optimization model problem

$$\begin{aligned}(QP)\ \inf\ & x^T Ax + 2a^T x + \alpha \\ s.t.\ & x^T B_i x + 2b_i^T x + \beta_i \leq 0,\ i = 1, \ldots, m \\ & Hx = d,\end{aligned}$$

where $A, B_i \in \mathbb{R}^{n\times n}$ are symmetric matrices, $a, b_i \in \mathbb{R}^n$ are vectors, $\alpha, \beta_i \in \mathbb{R}$, $i = 1, \ldots, m$, are scalars, $H \in \mathbb{R}^{k\times n}$ is a matrix and $d \in \mathbb{R}^k$. Model problems of the form (QP) cover a broad range of quadratic optimization problems and, in particular, include trust-region problems and quadratic programming problems [8,11,18,20–22,26,28,30].

The standard Lagrangian dual problem for (QP) is given by

$$(LDP)\ \max_{(\lambda,\mu)\in\mathbb{R}^m_+\times\mathbb{R}^k}\ \inf_{x\in\mathbb{R}^n} f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \sum_{j=1}^k \mu_j \left(h_j^T x - d_j\right),$$

where $f(x) = x^T Ax + 2a^T x + \alpha$, $g_i(x) = x^T B_i x + 2b_i^T x + \beta_i$, $i = 1, \ldots, m$, $H = [h_1, \ldots, h_k]^T$ and $h_j \in \mathbb{R}^n$, $j = 1, \ldots, k$. For (QP), Lagrangian duality means that the optimal values of (QP) and (LDP) are equal and (LDP) attains its maximum, that is, $\text{argmax(LDP)} \neq \emptyset$.

Research on Lagrangian duality for quadratic optimization has so far been limited mainly to problems (QP) without the linear equality constraints. In this case, Lagrangian duality has been established for various classes of non-convex quadratic optimization problems including trust-region problems [1–3,5,17,24,27,29,30]. In particular, the Slater constraint qualification is generally used to guarantee Lagrangian duality for trust-region problems.

Unfortunately, Lagrangian duality often fails even for linearly constrained trust-region problems under the Slater constraint qualification (SCQ) that $Hx_0 = d$, $x_0^T B_1 x_0 + 2b_1^T x_0 + \beta_1 < 0$, for some $x_0 \in \mathbb{R}^n$. For instance, consider the trust-region problem with a linear constraint

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \{-x_2^2 \mid x_1^2 + x_2^2 \leq 1,\ x_1 + x_2 = 0\}.$$

Its (global) optimal value $-\frac{1}{2}$ is attained at $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, whereas its Lagrangian dual problem

$$\max_{\lambda\geq 0,\mu\in\mathbb{R}}\ \inf_{(x_1,x_2)\in\mathbb{R}^2} -x_2^2 + \lambda(x_1^2 + x_2^2 - 1) + \mu(x_1 + x_2)$$

has -1 as its optimal value, and so, Lagrangian duality fails, but the Slater condition holds for the trust-region problem.

However, the following regularized Lagrangian dual problem

$$\max_{\lambda\geq 0}\ \inf_{x_1+x_2=0} -x_2^2 + \lambda(x_1^2 + x_2^2 - 1)$$

attains its optimal value $-\frac{1}{2}$ at $\lambda = 1/2$. Thus, the optimal values of the trust-region problem and its regularized dual [10,19] problem are equal. Note that the equality constraint is not relaxed in the regularized dual problem.

The aim of this paper is two fold. The first is that we establish Lagrange multiplier conditions which characterize a *regularized Lagrangian duality* in the sense that the optimal values of (QP) and its regularized Lagrangian dual

$$(DP) \quad \max_{\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}^m_+} \inf_{x\in H^{-1}(d)} \{f(x) + \sum_{i=1}^{m} \lambda g_i(x)\}$$

are equal and the regularized dual (DP) attains its maximum, where $H^{-1}(d) := \{x : Hx = d\}$. Obviously, in the absence of the linear equality constraints, the regularized dual (c.f. [10,19]) of (QP) collapses to the standard Lagrangian dual, and so, our regularized duality characterizations extend corresponding known results of Lagrangian duality [23,28,29].

The second aim is that we derive practical conditions characterizing the regularized duality for the important class of (QP) with a single quadratic inequality constraint. This class of problems includes trust-region problems with linear equality constraints. They often appear in the form of optimization sub-problems when solving nonlinear equality constrained optimization problems by trust-region techniques [8,28,30].

The paper makes two key contributions. We show that the positive semidefinitness of the Hessian of the Lagrangian, $A + \sum_{i=1}^{m} \lambda_i B_i$, over $\ker(H)$, characterizes the regularized Lagrangian duality for (QP), where $\lambda_i' s$ are the Lagrange multipliers associated with a minimizer of (QP) and $\ker(H)$ is the kernel of H, given by $\ker(H) = \{x \in \mathbb{R}^n : Hx = 0\}$.

The second contribution is that, in the case of (QP) with a single quadratic inequality constraint such as the trust-region problems with linear equality constraints, we show that (SCQ) is necessary and sufficient for the regularized Lagrangian duality in the sense that the regularized duality holds for each quadratic objective function over the constraints if and only if (SCQ) holds. Related characterizations for convex programming problems can be found in [13].

Also, we provide classes of non-convex quadratic optimization problems, including a class of important CDT-subproblems [1,5] involving linear equality constraints, where (SCQ) ensures regularized Lagrangian duality.

Our approach is to extract required convexity from the quadratic programs (QP) to derive the duality results and also to exploit the hidden convexity of quadratic forms given by Dines' theorem [9] to examine (QP) with a single quadratic inequality constraint. The characterization of regularized duality for trust-region problems is achieved by establishing a new theorem of the alternative for systems involving two quadratic inequalities and finitely many linear equalities. Our theorem of the alternative generalizes the corresponding theorem of Yuan [25,30] for two quadratic inequalities. Recently, various theorems of the alternative for quadratic inequality systems have been given in [15].

2 Regularized duality and linear equalities

In this Section, we present a duality theorem for the dual pair (QP) and its regularized Lagrangian dual (DP), extending the corresponding Lagrangian duality for quadratic problems with only quadratic inequality constraints [1,23,28,29].

We begin by fixing the notation and definitions that will be used later in the paper. The real line is denoted by $\mathbb{R}$ and the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The dimension of a subspace C is denoted by $\dim(C)$. The set of all non-negative vectors of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, and the interior of $\mathbb{R}^n_+$ is denoted by $\text{int}\mathbb{R}^n_+$. The space of all $(n \times n)$ symmetric matrices is denoted by S^n. The $(n \times n)$ identity matrix is denoted by I_n. The notation $A \succeq B$ means that the matrix $A - B$ is positive semidefinite. Moreover, the notation $A \succ B$ means the matrix $A - B$ is positive definite.

For the model problem (QP), we let $f(x) = x^T Ax + 2a^T x + \alpha$ and $g_i(x) = x^T B_i x + 2b_i^T x + \beta_i$, $i = 1, \ldots, m$, and assume that the feasible set $F := \{x : g_i(x) \le 0,\ i =$

$1, \ldots, m, Hx = d\} \neq \emptyset$. Denote $H = [h_1, \ldots, h_k]^T$ where each $h_j \in \mathbb{R}^n$, $j = 1, \ldots, k$. Denote the optimal values of (QP) and (DP) by $v(QP)$ and $v(DP)$. It is easy to check that the weak duality $v(QP) \geq v(DP)$ holds. Recall that $\overline{x} \in F$ is called a KKT point of (QP) with $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ if there exists $w \in \mathbb{R}^k$ such that

$$\nabla\left(f + \sum_{i=1}^{m} \lambda_i g\right)(\overline{x}) + H^T w = 0 \text{ and } \lambda_i g_i(\overline{x}) = 0,\ i = 1, \ldots, m. \tag{2.1}$$

The corresponding λ satisfying (2.1) is called the KKT multiplier associated with $\overline{x}$.

Theorem 2.1 *For the dual pair (QP) and (DP), suppose that* $\operatorname{argmin}(QP) \neq \emptyset$. *Then, the following two statements are equivalent:*

(i) $v(QP) = v(DP)$ *and* $\operatorname{argmax}(DP) \neq \emptyset$*;*
(ii) $A + \sum_{i=1}^m \lambda_i B_i$ *is positive semidefinite on* $\ker(H)$, *for some KKT multiplier $\lambda \in \mathbb{R}^m_+$ associated with a feasible point $\overline{x}$ of (QP).*

Proof [(i) $\Rightarrow$ (ii)] Let $\overline{x} \in \operatorname{argmin}(QP)$. Then, by the assumption, $v(DP) = f(\overline{x})$ and so, there exists $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ such that, for each $x \in H^{-1}(d)$,

$$f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) - f(\overline{x}) \geq 0.$$

This gives us that $\sum_{i=1}^m \lambda_i g_i(\overline{x}) \geq 0$ and, by the feasibility of $\overline{x}$, $\lambda_i g_i(\overline{x}) = 0$. This implies that $\overline{x}$ is a global minimizer of $f + \sum_{i=1}^m \lambda_i g_i$ over $H^{-1}(d)$. Letting $L(x) = f(x + \overline{x}) + \sum_{i=1}^m \lambda_i g(x + \overline{x})$, we see that for all $x \in \ker(H)$, $L(x) \geq L(0)$. So, there exists $w \in \mathbb{R}^k$ such that

$$\nabla\left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(\overline{x}) + H^T w = \nabla L(0) + H^T w = 0.$$

We now show that $A + \sum_{i=1}^m \lambda_i B_i$ is positive semidefinite on $\ker(H)$. To see this, note from the Taylor expansion that, for any $v \in \ker(H)$,

$$\begin{aligned} 0 &\leq \left(f(\overline{x} + v) + \sum_{i=1}^{m} \lambda_i g_i(\overline{x} + v)\right) - \left(f(\overline{x}) + \sum_{i=1}^{m} \lambda_i g_i(\overline{x})\right) \\ &= \nabla\left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(\overline{x})^T v + v^T\left(A + \sum_{i=1}^{m} \lambda_i B_i\right) v \\ &= v^T\left(A + \sum_{i=1}^{m} \lambda_i B_i\right) v, \end{aligned}$$

where the last equality holds as $\nabla(f + \sum_{i=1}^m \lambda_i g_i)(\overline{x})^T v = -(H^T w)^T v = w^T H v$ and $v \in \ker(H)$. Hence, (ii) holds.

[(ii) $\Rightarrow$ (i)] Since $\overline{x}$ is a KKT point with $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$, there exists $w \in \mathbb{R}^k$ such that $\nabla(f + \sum_{i=1}^m \lambda_i g)(\overline{x}) + H^T w = 0$, $\lambda_i g_i(\overline{x}) = 0$. Then for each feasible x_0 of (QP),

$$\begin{aligned} &\left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(x_0) - \left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(\overline{x}) \\ &= \nabla\left(f + \sum_{i=1}^{m} \lambda_i g_i\right)(\overline{x})^T (x_0 - \overline{x}) + \frac{1}{2}(x_0 - \overline{x})^T\left(A + \sum_{i=1}^{m} \lambda_i B_i\right)(x_0 - \overline{x}) \geq 0, \end{aligned}$$

as $A + \sum_{i=1}^{m} \lambda_i B_i$ is positive semidefinite on $\ker(H)$, $H(x_0 - \overline{x}) = 0$ and $\nabla(f + \lambda g)(\overline{x})^T (x_0 - \overline{x}) = -w^T H(x_0 - \overline{x}) = 0$. This gives us that for each feasible x_0 of (QP),

$$f(x_0) + \sum_{i=1}^{m} \lambda_i g_i(x_0) \geq f(\overline{x}) + \sum_{i=1}^{m} \lambda_i g_i(\overline{x}) = f(\overline{x}),$$

which, in turn, yields $f(x_0) \geq f(\overline{x}) - \sum_{i=1}^{m} \lambda_i g_i(x_0) \geq f(\overline{x})$. Hence, $\overline{x}$ is a global minimizer of (QP), and $v(QP) = v(DP)$ with $\lambda \in \operatorname{argmax}(DP)$. □

We now provide an example of a non-convex quadratic program with a non-convex quadratic inequality constraint which enjoys regularized Lagrangian duality and verifies Theorem 2.1. However, Lagrangian duality fails.

Example 2.1 Consider the quadratic program

$$(E_1) \quad \min_{(x_1,x_2)\in\mathbb{R}^2} -x_2^2 - 2x_1x_2 \quad \text{s.t.} \quad 1 - x_1^2 \leq 0,\ -1 - x_2 \leq 0,\ x_1 + x_2 = 1.$$

Observe that the problem (E_1) is of the form (QP) where $n = 2$, $H = (1, 1)$, $d = 1$, $a = b_1 = (0, 0)^T$, $b_2 = (0, -1/2)^T \alpha = 0$, $\beta_1 = 1$, $\beta_2 = -1$

$$A = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix},\ B_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

By eliminating x_2 using the equality constraint, the problem (E_1) becomes

$$\min\{x_1^2 - 1 \mid x_1^2 \geq 1,\ x_1 \leq 2\}$$

which has global minimizers $x_1 = -1$ and $x_1 = 1$. So, the global minimizers of (E_1) are $\overline{x} = (1, 0)$ and $\overline{z} = (-1, 2)$, and the optimal value $v(E_1) = 0$.

The Lagrangian dual of (E_1) becomes

$$(LDE_1) \quad \max_{\lambda_1,\lambda_2\geq 0,\mu\in\mathbb{R}} \inf\{-x_2^2 - 2x_1x_2 + \lambda_1(1 - x_1^2) + \lambda_2(-1 - x_2) + \mu(x_1 + x_2 - 1)\}.$$

Note that, for any $\lambda_1, \lambda_2 \geq 0$ and for any $\mu \in \mathbb{R}$,

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} \{-x_2^2 - 2x_1x_2 + \lambda_1(1 - x_1^2) + \lambda_2(-1 - x_2) + \mu(x_1 + x_2 - 1)\} = -\infty.$$

Thus $v(LDE_1) = -\infty$. Therefore, Lagrangian duality fails. The regularized dual problem of (E_1) can be formulated as

$$(DE_1) \max_{\lambda_1,\lambda_2\geq 0} \inf_{x_1+x_2=1} \{-x_2^2 - 2x_1x_2 + \lambda_1(1 - x_1^2) + \lambda_2(-1 - x_2)\}.$$

Then, it can easily be verified that $v(E_1) = v(DE_1)$ and (DE) attains its maximum at $(\lambda_1, \lambda_2) = (1, 0)$.

On the other hand, for $\mu = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$,

$$A + \lambda_1 B_1 + \lambda_2 B_2 = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix},$$

$v^T(A + \lambda_1 B_1 + \lambda_2 B_2)v = 0$ when $v = (v_1, v_2)$ with $v_1 + v_2 = 0$, and

$$\nabla(f + \lambda_1 g_1 + \lambda_2 g_2)(\overline{x}) + H^T\mu = (A + \lambda_1 B_1)\overline{x} + H^T\mu = (0, 0)^T \text{ and } \lambda_1 g(\overline{x}) = 0.$$

So, statement (ii) of Theorem 2.1 is satisfied. Note, however, that the Hessian of the Lagrangian corresponding to the global minimizer $\overline{x}$, $A + \lambda_1 B_1 + \lambda_2 B_2$, is not positive semidefinite.

It is worth noting that if the standard Lagrangian duality holds then the regularized Lagrangian duality also holds as $v(QP) \geq v(DP) \geq v(LDP)$. We now see that in the absence of linear equality constraints in (QP), Theorem 2.1 yields necessary and sufficient conditions for Lagrangian duality.

Consider the special case for (QP) without the linear equality constraints:

$$\begin{aligned}(QP_0)\ \inf\ & x^T Ax + 2a^T x + \alpha \\ s.t.\ & x^T B_i x + 2b_i^T x + \beta_i \leq 0,\ i = 1, \ldots, m.\end{aligned} \tag{2.2}$$

Its Lagrangian dual problem can be stated as follows:

$$(LDP_0)\ \max_{\lambda=(\lambda_1,\ldots,\lambda_m)\in\mathbb{R}^m_+} \inf_{x\in\mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda g_i(x) \right\}. \tag{2.3}$$

Corollary 2.1 *Suppose that* argmin$(QP_0) \neq \emptyset$. *Then, the following two statements are equivalent:*

(i) $v(QP_0) = v(LDP_0)$ *and* argmax$(LDP_0) \neq \emptyset$*;*
(iii) $A + \sum_{i=1}^m \lambda_i B_i \succeq 0$*, for some KKT multiplier* $\lambda \in \mathbb{R}^m_+$ *associated with a feasible point* $\overline{x}$ *of* (QP_0).

Proof The conclusion follows from Theorem 2.1 by letting $H = 0$ and $d = 0$. □

3 Linearly constrained trust-region problems

In this Section, we establish that the regularized Lagrangian duality is characterized in terms of the Slater constraint qualification for the general trust-region model problem with linear equality constraints [8,28]

$$\begin{aligned}(QP_f)\ \inf\ & f(x) \\ s.t.\ & x^T B_1 x + 2b_1^T x + \beta_1 \leq 0, \\ & Hx = d,\end{aligned} \tag{3.4}$$

where f is a quadratic function on $\mathbb{R}^n$ and $g_1(x) = x^T B_1 x + 2b_1^T x + \beta_1$ and $B_1 \in S^n$ is not necessarily positive semidefinite. Its dual problem (DP_f) can be formulated as follows

$$(DP_f)\ \max_{\lambda\geq 0} \inf_{x\in H^{-1}(d)} \{f(x) + \lambda g_1(x)\}. \tag{3.5}$$

The values of these problems (QP_f) and (DP_f) are denoted respectively by $v(QP_f)$ and $v(DP_f)$.

An alternative theorem for quadratic systems involving linear equalities plays a key role in deriving the duality characterization. To establish this alternative theorem, we recall the following well known Dines' Theorem on the joint-range convexity of homogeneous quadratic functions.

Lemma 3.1 (Dines' Theorem [9,25]) *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$ *be defined by* $f(x) = x^T Ax$ *and* $g(x) = x^T Bx$*, where* $A, B \in S^n$*. Then the set* $\{(x^T Ax, x^T Bx) : x \in \mathbb{R}^n\}$ *is convex.*

Theorem 3.1 (Alternative Theorem) *Let* $H \in \mathbb{R}^{k\times n}$ *and* $d \in \mathbb{R}^k$. *Let the quadratic functions* $f, g_1 : \mathbb{R}^n \to \mathbb{R}$ *be given by* $f(x) = x^T Ax + 2a^T x + \alpha$ *and* $g_1(x) = x^T B_1 x + 2b_1^T x + \beta_1$, *where* $A, B_1 \in S^n$, $a, b_1 \in \mathbb{R}^n$, $\alpha, \beta_1 \in \mathbb{R}$. *Then, exactly one of the following two statements holds:*

(i) $(\exists x \in \mathbb{R}^n)\ Hx = d,\ g_1(x) < 0$ *and* $f(x) < 0$
(ii) $(\exists(\lambda_0, \lambda_1) \in \mathbb{R}^2_+\backslash\{(0,0)\})(\forall x \in H^{-1}(d))\ \lambda_0 f(x) + \lambda_1 g_1(x) \geq 0.$

Proof [(i) $\Rightarrow$ Not(ii)] Suppose, on the contrary, that (ii) holds. Then, there exists $(\lambda_0, \lambda_1) \in \mathbb{R}^2_+\backslash\{(0,0)\}$ such that for all $x \in H^{-1}(d)$

$$\lambda_0 f(x) + \lambda_1 g_1(x) \geq 0. \tag{3.6}$$

From (i), there exists $x_0 \in \mathbb{R}^n$ with $Hx_0 = d$, $g_1(x_0) < 0$ and $f(x_0) < 0$. Since $(\lambda_0, \lambda_1) \in \mathbb{R}^2_+\backslash\{(0,0)\}$, it follows that

$$\lambda_0 f(x_0) + \lambda_1 g_1(x_0) < 0.$$

This contradicts (3.6).

[Not(i) $\Rightarrow$ (ii)] Assume that the following system has no solution

$$Hx = d,\ g_1(x) < 0 \text{ and } f(x) < 0. \tag{3.7}$$

To see (ii), define the following homogeneous functions $\widetilde{f}, \widetilde{g}_1 : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\widetilde{f}(x,\rho) = x^T Ax + 2\rho a^T x + \rho^2\alpha = \begin{pmatrix} x \\ \rho \end{pmatrix}^T \begin{pmatrix} A & a \\ a^T & \alpha \end{pmatrix} \begin{pmatrix} x \\ \rho \end{pmatrix} = \widetilde{x}^T \widetilde{A}\widetilde{x} \tag{3.8}$$

and

$$\widetilde{g}_1(x,\rho) = x^T B_1 x + 2\rho b_1^T x + \rho^2\beta_1 = \begin{pmatrix} x \\ \rho \end{pmatrix}^T \begin{pmatrix} B_1 & b_1 \\ b_1^T & \beta_1 \end{pmatrix} \begin{pmatrix} x \\ \rho \end{pmatrix} = \widetilde{x}^T \widetilde{B}\widetilde{x}, \tag{3.9}$$

where $\widetilde{x} = \begin{pmatrix} x \\ \rho \end{pmatrix}$, $\widetilde{A} = \begin{pmatrix} A & a \\ a^T & \alpha \end{pmatrix}$ and $\widetilde{B}_1 = \begin{pmatrix} B_1 & b_1 \\ b_1^T & \beta_1 \end{pmatrix}$.

Define the vector valued function $\tilde{h} : \mathbb{R}^{n+1} \to \mathbb{R}^k$ by

$$\widetilde{h}(x,\rho) = Hx - \rho d = \begin{pmatrix} H & -d \end{pmatrix} \begin{pmatrix} x \\ \rho \end{pmatrix}. \tag{3.10}$$

Then, the following system has no solution

$$\tilde{h}(x,\rho) = 0,\ \tilde{g}(x,\rho) < 0 \text{ and } \tilde{f}(x,\rho) < 0. \tag{3.11}$$

Otherwise, there exists $(x_0, \rho_0) \in \mathbb{R}^n \times \mathbb{R}$ such that $\tilde{h}(x_0, \rho_0) = 0$, $\tilde{g}_1(x_0, \rho_0) < 0$ and $\tilde{f}(x_0, \rho_0) < 0$. If $\rho_0 \neq 0$, then $g_1(\frac{x_0}{\rho_0}) = \rho_0^{-2}\tilde{g}_1(x_0, \rho_0) \leq 0$, $H(\frac{x_0}{\rho_0}) - d = \rho_0^{-1}\tilde{h}(x_0, \rho_0) = 0$ and $f(\frac{x_0}{\rho_0}) = \rho_0^{-2}\tilde{f}(x_0, \rho_0) < 0$ which contradicts the assumption that the system (3.7) has no solution.

If $\rho_0 = 0$, then there exists some $x_0 \in \mathbb{R}^n$ such that $Hx_0 = 0$, $x_0^T B_1 x_0 < 0$ and $x_0^T Ax_0 < 0$. Now, fix $\overline{x} \in \mathbb{R}^n$ with $H\overline{x} = d$. Then, for each $t \in \mathbb{R}$,

$$f(tx_0 + \overline{x}) = t^2 x_0^T Ax_0 + 2t(a + A\overline{x})^T x_0 + f(\overline{x}),$$
$$g_1(tx_0 + \overline{x}) = t^2 x_0^T B_1 x_0 + 2t(b_1 + B_1\overline{x})^T x_0 + g_1(\overline{x})$$

and

$$H(tx_0 + \overline{x}) = H\overline{x} = d.$$

Since $x_0^T B_1 x_0 < 0$ and $x_0^T A x_0 < 0$, for all $t > 0$ large enough one has $f(tx_0 + \overline{x}) < 0$ and $g_1(tx_0 + \overline{x}) < 0$. This implies that the following system has a solution

$$H(x) = d,\ g_1(x) < 0 \text{ and } f(x) < 0.$$

This again contradicts the assumption that the system (3.7) has no solution.

Now, we see that the system $\tilde{f}(x,\rho) < 0, \tilde{g}_1(x,\rho) < 0$, has no solution $(x,\rho) \in K$, where $K := \{(x,\rho) \in \mathbb{R}^{n+1} : \tilde{h}(x,\rho) = 0\}$. So, $\Omega \cap (-\text{int}\mathbb{R}^2_+) = \emptyset$, where $\Omega := \{(\tilde{f}(x,\rho), \tilde{g}_1(x,\rho)) : (x,\rho) \in K\}$. Let the dimension of the subspace K be m. Then, we can find a matrix $Q \in \mathbb{R}^{(n+1)\times m}$ of full rank such that $K = \{Qy : y \in \mathbb{R}^m\}$. This gives us that

$$\begin{aligned}\Omega &= \{(\widetilde{x}^T \widetilde{A}\widetilde{x}\ ,\ \widetilde{x}^T \widetilde{B}_1\widetilde{x}) : \widetilde{x} \in K\}\\ &= \{(y^T(Q^T\widetilde{A}Q)y\ ,\ y^T(Q^T\widetilde{B}_1Q)y)\ :\ y \in \mathbb{R}^m\}.\end{aligned}$$

So, from Dines' theorem, Ω is a convex set in $\mathbb{R}^2$.

Applying the Hahn–Banach separation theorem, we get $(\lambda_0, \lambda_1) \in \mathbb{R}^2_+ \backslash \{(0,0)\}$ such that for all $(x,\rho) \in K$,

$$\lambda_0 \tilde{f}(x,\rho) + \lambda_1 \tilde{g}_1(x,\rho) \geq 0. \tag{3.12}$$

Setting $\rho = 1$ and, noting that $\tilde{f}(x,1) = f(x), \tilde{g}_1(x,1) = g_1(x)$ and $\{x : Hx = d\} = \{x : (x,1) \in K\}$, we have $\lambda_0 f(x) + \lambda_1 g_1(x) \geq 0$ whenever $Hx = d$. □

In passing observe that Theorem 3.1 extends the corresponding alternative theorem of Yuan [30] where $H = 0$ and $d = 0$. We would like to point out that Theorem 3.1 may also be derived from the results of [15]. However, we have given a direct and self-contained proof for Theorem 3.1.

In the following we establish that the Slater condition is necessary and sufficient for regularized duality of trust-region problems with linear equality constraints, extending the corresponding result in [14] for the standard trust-region problems without the linear equality constraints. We first show this result in the case of a homogeneous linear equality constraints.

Let us recall some basic tools of convex analysis. The set $K \subset \mathbb{R}^n$ is a cone if $\lambda K \subset K$, for each $\lambda \geq 0$. Let C be a closed and convex subset of $\mathbb{R}^n$ and let $\overline{x} \in C$. The *normal cone* of C at $\overline{x}$ is defined as $N_C(\overline{x}) := \{y \in \mathbb{R}^n\ :\ y^T(x - \overline{x}) \leq 0 \text{ for all } x \in C\}$. If C is a subspace of $\mathbb{R}^n$ then the *orthogonal complement* of C is given by $C^\perp := \{y \in \mathbb{R}^n\ :\ y^T x = 0, \forall x \in C\}$. Moreover, for a convex function f on $\mathbb{R}^n$, the *subdifferential* of f [4,16] at a point $\overline{x}$ is defined by $\partial f(\overline{x}) := \{y : y^T(x - \overline{x}) \leq f(x) - f(\overline{x}) \text{ for all } x \in \mathbb{R}^n\}$.

Theorem 3.2 *Let $H \in \mathbb{R}^{k\times n}$. Let $g_1 : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function not identically zero on* $\ker(H)$. *Suppose that $F := \{x : g_1(x) \leq 0,\ Hx = 0\} \neq \emptyset$. Then, the following statements are equivalent:*
(i) $\exists x_0 \in \mathbb{R}^n$, $Hx_0 = 0$, $g_1(x_0) < 0$
(ii) *For each quadratic function f,*

$$\inf_{\substack{g1(x)\leq 0\\ Hx=0}} f(x) = \inf_{Hx=0}\{f(x) + \lambda g_1(x)\},$$

for some $\lambda \geq 0$.

Proof [(i) $\Rightarrow$ (ii)] Let f be an arbitrary quadratic function. Without loss of generality, we may assume that $\mu := \inf_{x\in F} f(x)$ is finite. Then, $[g_1(x) \leq 0,\ Hx = 0 \Rightarrow\ f(x) - \mu \geq 0]$.

So, the system

$$Hx = 0,\ g_1(x) < 0 \text{ and } f(x) - \mu < 0,$$

has no solution. By Theorem 3.1, there exists $(\lambda_0, \lambda_1) \in \mathbb{R}^2_+ \backslash \{(0,0)\}$ such that $\lambda_0(f(x) - \mu) + \lambda_1 g_1(x) \geq 0$ for all $x \in \ker(H)$. We can assume that $\lambda_0 \neq 0$. Otherwise, $\lambda_1 > 0$ and $\lambda_1 g_1(x) \geq 0$ for all $x \in \ker(H)$. This implies that $g_1(x) \geq 0$ for all $x \in \ker(H)$ which contradicts (i). So, for each $x \in \ker(H)$, $f(x) + \lambda g_1(x) \geq \mu$, where $\lambda = \frac{\lambda_1}{\lambda_0}$. This implies that

$$\mu \leq \inf_{x \in \ker(H)} \{f(x) + \lambda g_1(x)\}.$$

Note that $\mu \geq \inf_{x \in \ker(H)}\{f(x) + \lambda g_1(x)\}$ always holds. Hence, (ii) holds.

[(ii) $\Rightarrow$ (i)] To see (i), we proceed by the method of contradiction and suppose that $g_1(x) := x^T B_1 x + 2b_1^T x + \beta_1 \geq 0$ whenever $x \in \ker(H)$. This implies that B_1 is positive semidefinite on $\ker(H)$. [Otherwise, there exists $v \in \ker(H)$ such that $v^T Bv < 0$. Take $a_0 \in F$. Since $d = 0$, $a_0 \in \ker(H)$. So, $\lim_{t\to\infty} g_1(a_0 + tv) \to -\infty$ and $a_0 + tv \in \ker(H)$ for all $t \in \mathbb{R}$. This is a contradiction.] Then, $g := g_1 + \delta_{\ker(H)}$ is a nonnegative lower semicontinuous convex function which is not identically zero and

$$\{x : g(x) = 0\} = \{x : g(x) \leq 0\} = \ker(H) \cap \{x : g_1(x) \leq 0\},$$

where $\delta_{\ker(H)}$ is the indicator function of $\ker(H)$. Since g is convex and it attains its minimum at each point of $\{x : g(x) \leq 0\}$ and

$$\partial g(x) = 2B_1 x + 2b + \ker(H)^\perp, \tag{3.13}$$

it then follows that

$$\{x : g(x) \leq 0\} = \{x : 0 \in \partial g(x)\} = \{x : 2B_1 x + 2b \in \ker(H)^\perp\}. \tag{3.14}$$

On the other hand, using (ii), we now show that

$$N_{\{x:g(x)\leq 0\}}(\overline{x}) \subseteq \ker(H)^\perp. \tag{3.15}$$

where $\overline{x} \in \{x : g(x) \leq 0\}$. To see this, let $y \in N_{\{x:g(x)\leq 0\}}(\overline{x})$. Then $y^T(x - \overline{x}) \leq 0$ whenever $g(x) \leq 0$, and so,

$$g(x) \leq 0 \Rightarrow -y^T x \geq -y^T \overline{x}.$$

Then, by (ii) with $f(x) = -y^T(x - \overline{x})$, there exists $\lambda \geq 0$ such that $-y^T(x - \overline{x}) + \lambda g(x) \geq 0$, for all $x \in \mathbb{R}^n$. As $g(\overline{x}) = 0$, we obtain that $y \in \partial(\lambda g)(\overline{x}) = \lambda \partial g(\overline{x})$ and hence

$$N_{\{x:g(x)\leq 0\}}(\overline{x}) \subseteq \bigcup_{\lambda \geq 0} \{\lambda \partial g(\overline{x})\}. \tag{3.16}$$

Since $\overline{x} \in \{x : g(x) = 0\}$ (as g is nonegative), (3.13) and (3.14) imply that

$$\partial g(\overline{x}) = 2B_1 \overline{x} + 2b_1 + \ker(H)^\perp \subseteq \ker(H)^\perp.$$

So,

$$N_{\{x:g(x)\leq 0\}}(\overline{x}) \subseteq \bigcup_{\lambda \geq 0} \{\lambda \partial g(\overline{x})\} \subseteq \ker(H)^\perp.$$

Thus (3.15) holds.

Now, as $\overline{x} \in \{x : g(x) \leq 0\}$, it follows from (3.14) that

$$\{x : g(x) \leq 0\} = \{x : 2B_1x + 2b \in \ker(H)^{\perp}\} \\ = \overline{x} + S, \tag{3.17}$$

where S is a subspace defined by $S := \{x : 2B_1x \in \ker(H)^{\perp}\}$. Then, (3.15) and (3.17) give us that $\ker(H)^{\perp} \supseteq N_{\{x:g(x)\leq 0\}}(\overline{x}) = N_{\overline{x}+S}(\overline{x}) = S^{\perp}$. This implies that

$$\ker(H) \subseteq S = \{x : 2B_1x \in \ker(H)^{\perp}\},$$

and so, $x^T B_1 x = 0$ for all $x \in \ker(H)$. This together with our assumption $g_1(x) = x^T B_1 x + 2b_1^T x + \beta_1 \geq 0$ for all $x \in \ker(H)$ gives us that $2b_1^T x + \beta_1 \geq 0$ for all $x \in \ker(H)$. So, $b_1^T x = 0$, for all $x \in \ker(H)$ and $\beta_1 \geq 0$. Take $a \in F$. Then, $a \in \ker$(H) and so, $g_1(a) = a^T B_1 a + 2b_1^T a + \beta_1 = \beta_1 \leq 0$. Therefore, we have $\beta_1 = 0$ and so $g_1(x) \equiv 0$ whenever $x \in \ker(H)$. This contradicts our assumption. □

As a consequence of Theorem 3.2 we obtain a necessary and sufficient condition for regularized duality of trust-region problems (QP_f).

Theorem 3.3 (Duality Theorem) *Let $H \in \mathbb{R}^{k\times n}$ and $d \in \mathbb{R}^k$. Let $g_1 : \mathbb{R}^n \to \mathbb{R}$ be a quadratic function not identically zero on $\{x : Hx = d\}$. Suppose that $F := \{x : g_1(x) \leq 0,\ Hx = d\} \neq \emptyset$. Then, the following statements are equivalent:*
(i) $\exists x_0 \in \mathbb{R}^n$, $Hx_0 = d$, $g_1(x_0) < 0$
(ii) *For each quadratic objective function f with $v(QP_f)$ finite, $v(QP_f) = v(DP_f)$ and* $\operatorname{argmax}(DP_f) \neq \emptyset$.

Proof [(i) ⇒ (ii)] Let f be a quadratic function with $v(QP_f)$ finite. Then, the system

$$Hx = d,\ \ g_1(x) < 0 \text{ and } f(x) - v(QP_f) < 0$$

has no solution. So, by Theorem 3.1, there exists $(\lambda_0, \lambda_1) \in \mathbb{R}^2_+ \backslash \{(0, 0)\}$ such that $\lambda_0(f(x) - v(QP_f)) + \lambda_1 g_1(x) \geq 0$ for all $x \in H^{-1}(d)$. Then, $\lambda_0 \neq 0$. Otherwise $\lambda_1 g_1(x) \geq 0$ with $\lambda_1 \neq 0$. This contradicts (i). So, for each $x \in H^{-1}(d)$, $f(x) + \lambda g_1(x) \geq v(QP_f)$, where $\lambda = \frac{\lambda_1}{\lambda_0}$. This implies that

$$v(QP_f) \leq \inf_{x\in H^{-1}(d)} \{f(x) + \lambda g_1(x)\} \leq \max_{\lambda\geq 0} \inf_{x\in H^{-1}(d)} \{f(x) + \lambda g_1(x)\} = v(DP_f).$$

Note that $v(QP_f) \geq v(DP_f)$ always holds. Hence, (ii) holds.

[(ii) ⇒ (i)] Assume that $d \neq 0$. Otherwise (i) readily follows from Theorem 3.2. Take $a_0 \in F = \{x : g_1(x) \leq 0,\ Hx = d\}$. Define $\bar{g}_1$ by $\bar{g}_1(x) = g_1(x + a_0)$, $x \in \mathbb{R}^n$. Note that $\{x : Hx = d\} = a_0 + \ker(H)$. Then (ii) gives us that, for each quadratic function f,

$$\inf_{\substack{g_1(x)\leq 0\\ Hx=d}} f(x) = \inf_{Hx=d} \{f(x) + \lambda g_1(x)\}, \tag{3.18}$$

for some $\lambda \geq 0$. Let $\bar{f}(x) = f(x - a_0)$. Then, by applying (3.18) with $f = \bar{f}$, we obtain that

$$\begin{aligned}\inf_{\substack{g_1(x)\leq 0\\ Hx=d}} \bar{f}(x) &= \inf_{Hx=d} \{\bar{f}(x) + \lambda g_1(x)\} \\ &= \inf_{H(a_0+x)=d} \{\bar{f}(a_0 + x) + \lambda g_1(a_0 + x)\} \\ &= \inf_{Hx=0} \{f(x) + \lambda \bar{g}_1(x)\}.\end{aligned}$$

On the other hand,

$$\inf_{\substack{g_1(x)\le 0\\ Hx=d}} \bar{f}(x) = \inf_{\substack{g_1(a_0+x)\le 0\\ H(a_0+x)=d}} \bar{f}(a_0+x) = \inf_{\substack{\bar{g}_1(x)\le 0\\ Hx=0}} f(x).$$

So, for each quadratic function f,

$$\inf_{\substack{\bar{g}_1(x)\le 0\\ Hx=0}} f(x) = \inf_{Hx=0}\{f(x)+\lambda \bar{g}_1(x)\},$$

for some $\lambda \ge 0$. Now, from Theorem 3.2, there exists $\bar{x}_0 \in \ker(H)$ such that $g_1(\bar{x}_0 + a_0) = \bar{g}_1(x_0) < 0$. Therefore, (i) holds by letting $x_0 = \bar{x}_0 + a_0$. □

We also deduce from Theorem 3.2 a characterization of regularized Lagrangian duality for the following trust region problem with linear equality constraint (cf. [28]):

$$\begin{aligned} (TR_f)\ \inf\ & f(x) \\ s.t.\ & \|x\|^2 \le 1, \\ & Hx = d, \end{aligned}$$

where $f(x) = x^T A_f x + 2a_f^T x + \alpha_f$. Its regularized dual problem (DTR_f) can be stated as follows:

$$(DTR_f)\ \max_{\lambda\ge 0} \inf_{x\in H^{-1}(d)} \{f(x) + \lambda(\|x\|^2 - 1)\}.$$

Corollary 3.1 *Let $H \in \mathbb{R}^{k\times n}$ and $d \in \mathbb{R}^k$. Suppose that $F := \{x : \|x\|^2 \le 1,\ Hx = d\} \ne \emptyset$. Then, the following statements are equivalent:*

(i) $\exists x_0 \in \mathbb{R}^n$, $Hx_0 = d$ *and* $\|x_0\|^2 < 1$.
(ii) *For each quadratic objective function f with $v(TR_f)$ finite, it holds that $v(TR_f) = v(DTR_f)$ and* argmax$(DTR_f) \ne \emptyset$.
(iii) *For each quadratic objective function f, $A_f + \lambda I_n$ is positive semidefinite on* $\ker(H)$ *for some KKT multiplier $\lambda \in \mathbb{R}_+$ associated with a feasible point $\overline{x}$.*

Proof From Theorem 3.2, we see that (i) is equivalent to (ii). To show that (ii) is equivalent to (iii), note that, for each quadratic objective function f, argmin$(TR_f) \ne \emptyset$. Therefore, the equivalence of (ii) and (iii) follows from Theorem 2.1 with $m = 1$ and $g_1(x) = \|x\|^2 - 1$. □

4 Quadratic programs with regularized duality

In this section we derive classes of non-convex quadratic optimization problems for which the regularized duality holds under the Slater condition.

4.1 A class of CDT-problems with linear equalities

Consider the CDT model problem (cf [1,7]) with additional linear equalities

$$\begin{aligned} (QP_{CDT})\ \inf\ & x^T Ax + 2a^T x + \alpha \\ s.t.\ & \|Bx + b\|^2 \le \xi^2,\ \|x\|^2 \le \Delta^2, \\ & Hx = d, \end{aligned}$$

where $B \in \mathbb{R}^{m\times n}$, $A \in S^n$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^m$ and $\Delta, \xi \in [0, +\infty)$. Problems of this form arise when applying trust region techniques to solve nonlinear optimization problems with nonlinear equality constraints.

Its regularized dual problem becomes

$$(DP_{CDT}) \max_{\lambda=(\lambda_1,\lambda_2)\in\mathbb{R}^2_+} \inf_{x\in H^{-1}(d)} \left\{ f(x) + \lambda_1(\|Bx+b\|^2 - \xi^2) + \lambda_2(\|x\|^2 - \Delta^2)\right\}.$$

Corollary 4.1 *For (QP_{CDT}), suppose that A is positive semidefinite on* $\ker(H)$ *and that there exists $x_0 \in \mathbb{R}^n$ with $Hx_0 = d$ such that*

$$\|Bx_0 + b\|^2 < \xi^2 \text{ and } \|x_0\|^2 < \Delta^2.$$

Then, $v(QP_{CDT}) = v(DP_{CDT})$ and $\operatorname{argmax}(DP_{CDT}) \neq \emptyset$.

Proof Let $\overline{x} \in \operatorname{argmin}(QP_{CDT})$. Since Slater's condition holds and the feasible set is convex, $\overline{x}$ is a KKT point of (QP_{CDT}). Let $\lambda \in \mathbb{R}^2_+$ be the KKT multiplier associated with $\overline{x}$. Note that A is positive semidefinite on $\ker(H)$, and so, $A + \lambda_1 B^T B + \lambda_2 I_n$ is also positive semidefinite on $\ker(H)$. So, the conclusion follows from Theorem 2.1. □

The positive semidefinitness of A on $\ker(H)$ can be checked, for instance, by the following eigenvalue characterization [6]:

$[Hx = 0 \;\Rightarrow\; x^T Ax \geq 0]$ if and only if the matrix $\begin{pmatrix} A & H^T \\ H & 0 \end{pmatrix}$ has exactly q negative eigenvalues, where q is the rank of H.

4.2 A class of quadratic programs

Consider the following quadratic programming problem with equality constraints

$$\begin{aligned} (QPE)\ \inf\ & x^T Ax + 2a^T x + \alpha \\ s.t.\ & b_i^T x + \beta_i \leq 0,\ i = 1, \ldots, m \\ & Hx = d, \end{aligned}$$

where $A \in S^n$, $a, b_i \in \mathbb{R}^n$, $\alpha, \beta_i \in \mathbb{R}$, $i = 1, \ldots, m$, $H \in \mathbb{R}^{k\times n}$ and $d \in \mathbb{R}^k$.

Define $f(x) := x^T Ax + 2a^T x + \alpha$. Consider its associated regularized dual problem

$$(DPE) \max_{\lambda\in\mathbb{R}^m_+} \inf_{x\in H^{-1}(d)} \left\{ f(x) + \sum_{i=1}^m \lambda_i(b_i^T x + \beta_i)\right\}.$$

The values of these two problems are denoted by $v(QPE)$ and $v(DPE)$.

Theorem 4.1 *For (QPE), assume that* $\operatorname{argmin}(QPE) \neq \emptyset$. *Suppose that A is positive semidefinite on* $\ker(H)$. *Then, $v(QPE) = v(DPE)$ and* $\operatorname{argmax}(DPE) \neq \emptyset$.

Proof Let $\overline{x} \in \operatorname{argmin}(QPE)$ and let $F := \{x : b_i^T x + \beta_i \leq 0,\ i = 1, \ldots, m, Hx = d\}$. Since F is convex, it follows that

$$-\nabla f(\overline{x}) \in N_F(\overline{x}).$$

Note that (cf. [12, Proposition 2.2.2])

$$N_F(\overline{x}) \subseteq \left\{ \sum_{i=1}^m \lambda_i b_i + H^T w : \lambda_i \geq 0, w \in \mathbb{R}^k \right\}.$$

Thus, $\overline{x}$ is a KKT point of (QPE). Let $\lambda \in \mathbb{R}^m_+$ be the KKT multiplier associated with $\overline{x}$. Note that A is positive semidefinite on $\ker(H)$. So, the conclusion follows from Theorem 2.1 with $B_i = 0, i = 1, \ldots, m$. □

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